LUNEBERG LENSES FOR RAPID SCANNING RADAR ANTENNAS

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by

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PREFACE

This compilation of the data available on microwave application of variable index of refraction lenses was made at the United States Naval Postgraduate School during the latter half of the academic year 1953. It is an attempt to show the steps which have been undertaken to develop a lens which will permit focusing over a wide angle through movement of feed system alone.

The writer would like to express his appreciation to A. S. Dunbar of the Dalmo Victor Co., San Carlos, California, for his assistance in understanding the limitations of other types of antennas and to C. F. Klamm Jr. of the United States Naval Postgraduate School Faculty for his guidance in the preparation of this paper.

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SYMBOLS AND ABBREVIATIONS

a - plate spacing in lens

K - relative dielectric constant

n - index of refraction

r - radial distance from center

R - power reflection coefficient

s - arc length, measured from the axis of revolution

t_λ - thickness in free space wavelengths

T - power transmission coefficient

μ - relative loss

v - phase velocity

v_o - phase velocity of free space

λ - wavelength

SUMMARY

The purpose of this paper is to present in a concise form the limitations of conventional antennas and lenses, as now employed, in scanning over large areas by movement of the feed system relative to the antenna. In general such a system of scanning is required in modern radar systems for detection of aircraft as their speed is increased. The "perfect lens" as developed by Luneberg for optics is then presented as a solution, such a lens permitting focusing on the circumference of a spherical surface, provided the index of refraction varies as a function of the normalized radial distance ($n = \sqrt{2-r^2}$). The two dimensional microwave analogy of this theory has been tested by the construction of two lenses based on waveguide theory for variation of the index of refraction and the results of the tests are shown herein. Such tests proved the conformity in practice to theoretical prediction and showed the practicability of three dimensional lenses constructed of artificial dielectrics. The theory was then extended for the development of a lens which will reduce the feed circle size required.

CHAPTER I

MICROWAVE LENSES

With the increase of frequency used in radio systems, the wavelength has decreased to a point such that it is now possible to apply some of the principles of optics in the field of radio. The first practical applications utilized to focusing properties of reflectors, usually in the shape of paraboloids and parabolic cylinders, to focus the energy from either a point of line source into pencil beams. The possibility of focusing radio waves by means of lenses was realized in 1889 when Sir Oliver Lodge (15)* demonstrated the beaming effect of a lens constructed of pitch on the radiation from a spark gap oscillator. However no worthwhile endeavor was possible until convenient sources of power at centimetric wavelengths became available about 1940. Rust (22), in 1942, was the first to experiment by making use of a lens to correct the wave fronts emerging from an electromagnetic horn. Since then many types of lens antennas have been used in microwave transmissions.

According to J. Brown (2) there are two convenient classifications of lenses; one according to the function for which the lens is designed and the other is according to the nature of the lens material. The latter classification is composed of six sub-divisions. Namely:

1. Solid dielectrics
2. Metallic delay dielectrics
3. Metal plate dielectrics
4. Rodded dielectrics
5. Path length dielectrics
6. Miscellaneous dielectrics

*Numbers in parentheses refer to authors in bibliography

Solid dielectric lenses are analogous to optical lenses where the refractive index is related to the dielectric constant by the relation

$$n = \sqrt{K}$$

Some of the available solid dielectrics with their characteristics are:

Dielectric	Dielectric Constant	Loss Factor tan δ	Mechanical properties
Teflon	2.03	.00037	Flexible; difficult to join together and coat except under pressure
Polyethylene	2.25	<.005	Flexible, tough, can be welded together and mirror sprayed
Polystyrene	2.54	.00025 to .0016	Flexible, low shock resistance, can be cemented, low heat
Polystyrene foam	1.05	<.00003	Difficult to hold tolerances

The principal advantage of this type of lens is the simplicity and ease of manufacturing. The main disadvantage is the relative heaviness and attendant difficulty in supporting the structure.

Metallic delay dielectrics as developed by W. E. Kock (11) are sometimes referred to as artificial dielectrics wherein strips and disks of less than a wavelength in size behave basically as large scale molecules of a dielectric structure and give a refractive index greater than unity.

Metal plate lenses as described by W. E. Kock (12) are based on the fact that the phase velocity of a wave travelling between plates

is greater than that of free space provided the electric vector is parallel to the plates. In this case the refractive index may be defined as the ratio of the phase velocity in free space to the phase velocity between plates

$$n = \frac{v_o}{v}$$

Hence in metal plate lenses the refractive index is less than unity, a possibility which does not arise in optics.

Rodded dielectrics are similar to delay lenses in that they consist of a lattice structure of metal rods. The structure, however, produces a refractive index less than unity which varies with frequency. This is a disadvantage as compared with the delay lens, however rodded dielectrics are self supporting while delay dielectrics are not.

Path length lenses as developed by W. E. Kock (13) employ media, which may have an index of refraction either greater or less than unity, to alter the wave front by causing different portions of the wave to travel different distances, the velocity of propagation within the lens remaining constant at the free space value.

Miscellaneous lenses have one common feature; the refractive index within the lens varies from point to point. No practical method has been developed for accomplishing this in optics. However, by variable plate spacing or artificial dielectrics, the advantages obtained from variable refractive index may be applied in the design of microwave lenses. The application of the optical theories developed

by R. K. Luneberg (16) to microwave lenses fall into this classification.

Some of the main merits of lenses over other types of commonly used focusing antennas are (3):

1. An increase of 4 to 5 times in manufacturing tolerances as compared with a reflector system. According to J. D. Kraus (14) the allowable tolerances for an electrical path length arbitrarily set at 1/3th wavelength of various types of lens and reflector antennas are listed as below.

Type of Antenna	Type of tolerance	Amount of tolerance
Parabolic reflector	Surface contour	$\pm .03\lambda$
Dielectric lens (unzoned)	Thickness	$\pm \dfrac{.03\lambda}{n-1}$
	Index of refraction	$\pm \dfrac{.03\lambda}{nt_\lambda}$
Dielectric lens (zoned)	Thickness	$\pm .03\lambda$
	Index of refraction	$\pm .03 \dfrac{(n-1)}{n}$
E Plane metal plate lens (unzoned)	Thickness	$\pm .03 \dfrac{\lambda}{1-n}$
	Plate spacing	$\pm .03 \dfrac{\lambda}{(1-n^2)t_\lambda}$
E Plane metal plate lens (zoned)	Thickness	$\pm .03\lambda$
	Plate spacing	$\pm .03 \dfrac{n\lambda}{1 \neq n}$
H Plane metal plate lens	Length of path	$\pm .06\lambda$

2. Freedom from distortion of the pattern due to twisting or warping of the lens system within relatively wide limits provided the position of the source remains fixed.

3. The primary source is not in the path of the secondary beam and the scatter from the source does not arise so that reflection of power back into the feed can be reduced to very small proportions. This eases the problem of matching the feed to the oscillator.

4. A lens has two surfaces, allowing the designer two freedoms in dealing with it.

5. The level of the first side lobe in a well designed lens antenna will be approximately 22 db below that of the main beam, while that of the paraboloid rarely exceeds 16 db below the main beam.

Two effects to be considered in microwave antennas which are usually neglected in optics are spillover and surface reflections. Ideally all the energy radiated by the primary feed is focused by the lens, but in practice some of the energy will propogate around the edges of the lens. Energy lost in this way is known as spillover. A satisfactory compromise between spillover and utilizing maximum lens aperture is to employ a directive feed such that the distribution across the aperture is tapered to 10 db down at the edges as compared to amplitude at the center. With such a distribution, spillover is not excessive and side lobe level will not exceed about 22 db below level of the main beam.

In any type of lens which depends upon changes of refractive index for focusing, such discontinuities will cause reflection of power. The power reflection coefficient R defined as the ratio of the power reflected from any area of the interface to the power incident on the same area will approach in the limiting case of normal incidence the value

$$R = \left[\frac{n - 1}{n + 1}\right]^2$$

The power transmission coefficient T always has the value 1 - R. The above discussion applies only to solid dielectrics and modifications are necessary before they can be applied to the various artificial dielectrics used in lenses. These modifications as developed by J. Brown (3) take into account the many interfaces in strip delay and metal plate dielectrics. The work of C. Susskind (23) considers the action of obstacle type artificial dielectrics.

Lenses, in general, have aberrations corresponding to counterparts in optical lenses. These aberrations termed (a) defocusing, (b) spherical aberration, (c) coma, and (d) curvature of field are discussed in NRL Report R-3312 (17) and by A. S. Dunbar (5). In general, such aberrations will decrease the gain, increase the energy contained in the side lobes, broaden the beam width and in the case of coma, the development of a "coma lobe" which is unsymmetrical. Furthermore, they reduce the ability of the various lenses to be used as wide angle scanning antennas due to the fact that mass and size of antennas make it imperative the scanning be done by movement of the feed rather than the antenna. For example, a paraboloid of f/.5 has a scan arc of ±3 beamwidths while an f/1 has ±8 beamwidths, both limited by coma. A simple lens of f/1 has approximately a ±10 beamwidth scan and is limited by coma and astigmatism while a Schmidt lens of the same f number may be constructed to give as high as ±40 beamwidth scan before oblique spherical aberration causes excess deterioration.(6) One of the types of

lenses which are not limited by aberrations as to capability to scan by movement of feed alone is the Luneberg Lens.

CHAPTER II

THEORY OF LUNEBERG LENS

It was shown by Luneberg (16) that a plane wave illuminating a transparent sphere in which the refractive index is the correct function of the radial distance from the center will be brought to a focus at a point in the surface of the sphere. From symmetry, this point lies on the diameter normal to the plane of the wave. The function of radial distance which produces focusing on the circumference is

$$n = \sqrt{2 - \left[\frac{r}{R}\right]^2}$$

where R = radius of the sphere

or for a sphere of unit radius

$$n = \sqrt{2 - r^2}$$

A simplified derivation of the above expression is included in the appendix. If the sphere is immersed in a medium of refractive index n_o, a more general expression becomes

$$n = n_o \sqrt{2 - r^2}$$

Theoretically, a lens designed on the above principles would be a "perfect lens" since it would be free of all distortions and would have an unlimited arc of scan. Mechanical difficulties would attend any rapid scanning however, due to the large path the feed would have to follow if any appreciable aperture size was employed.

R. F. Rinehart (20) has attacked the problem by transforming the

plane variable refractive index system into a surface of revolution of constant refractive index one, whose equation

$$s = \frac{1}{2} \left(\rho + \sin^{-1} \rho \right)$$

becomes in cylindrical coordinates

$$z = \frac{1}{2}\left[4 - 3\rho^2 + 4(1-\rho^2)^{\frac{1}{2}}\right]^{\frac{1}{2}} - \frac{1}{\sqrt{3}} \log \left\{ \left[\frac{3}{2}\left\{1 + (1-\rho^2)^{\frac{1}{2}}\right\}\right]^{\frac{1}{2}} + \left[\frac{1}{2}\left\{3(1-\rho^2)^{\frac{1}{2}} + 1\right\}\right]^{\frac{1}{2}} \right\} - \frac{1}{2} + \frac{1}{2}\sqrt{3} \log (2 + \sqrt{3})$$

The surface has a horizontal tangent plane at $\rho = 0$ and a vertical tangent plane at $\rho = 1$. The surface is concave downward between $\rho = 0$ and $\rho = 1$, as drawn in Figure 1.

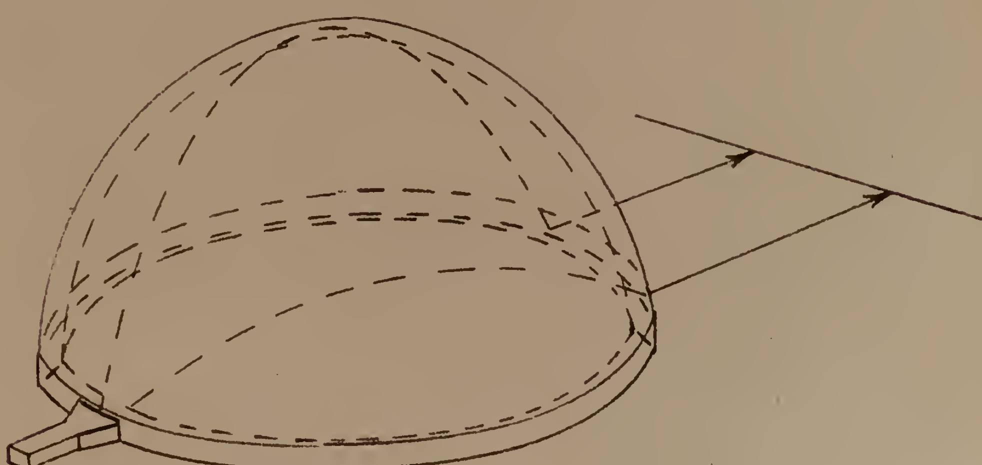

Figure 1.

The Luneberg Variable Refractive Index Transformation Into A Surface of Revolution of Constant Refractive Index

R. . Rinehart has further proposed two solutions to the problem of reducing feed circle diameter. One is the use of an elliptical reflector to product a virtual point source on the unit circle from an actual point source situated much closer, or even on the axis of the surface. Another solution (21) is the design of a surface of revolution such that the hat shaped surface and an annular region between $r = 1$ and $r = r_1$ will duplicate, with a refractive index of unity, the properties of the Luneberg system, as shown in Figure 2. The generating curve $r = r(s)$ is determined by the equations

$$ds/r = du/u$$

$$r = \mu u$$

where $$\mu = e^{w(r,\ r_1)}$$

and the function $w(r, r_1)$ is determined by

$$w(r, r_1) = \frac{1}{\pi} \int_r^1 \left(\arc\sin \frac{t}{r_1}\right) / \sqrt{t^2 - r^2}\, dt$$

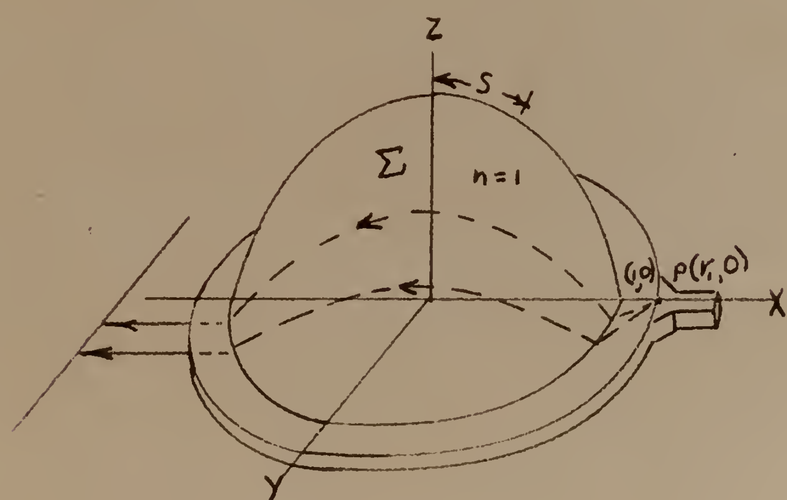

Figure 2.

The Curved Surface of Unit Radius with Attached Ring Lens Analogue of the Luneberg Lens

This surface requires an even larger feed circle $r = r_1$. However, it is possible to construct an equivalent system consisting of the same curved surface and an attached annular ring of outer radius unity and with a constant non-unit refractive index which amounts to turning the "hat brim" in. With any choice of feed circle radius r_0, there exists a corresponding refractive index for the annular ring of $n = 1/r_0$ which when attached to the surface of revolution of unit radius will give an equivalent Luneberg lens. The size of the feed circle therefore is limited only to the index of refraction and the problem of matching that index to the index of refraction of unity of the curved surface.

Another method of achieving a reduction in feed circle size applicable to the variable index of refraction type of lens is derived by extending the theory of Luneberg to permit the source to lie within the lens, as illustrated in Figure 3. This extension was developed by J. E. Eaton (7) and also by J. Brown (1). From this extension two cases in particular were studied. Case I is the result of holding the refractive index constant in the outer zone which, in order to satisfy the conditions, requires the index of refraction n_1, of the outer zone to be $1/r_0$, or in the case of the feed circle being one half the outer dimension $n_1 = 2$ and the variable index of refraction in the inner zone has a maximum value of 2.34 at the center of the lens and drops monotonically to 2 at $r = r_0$. The abrupt change of the index of refraction at the circumference would impart the same restrictions as in the case of the

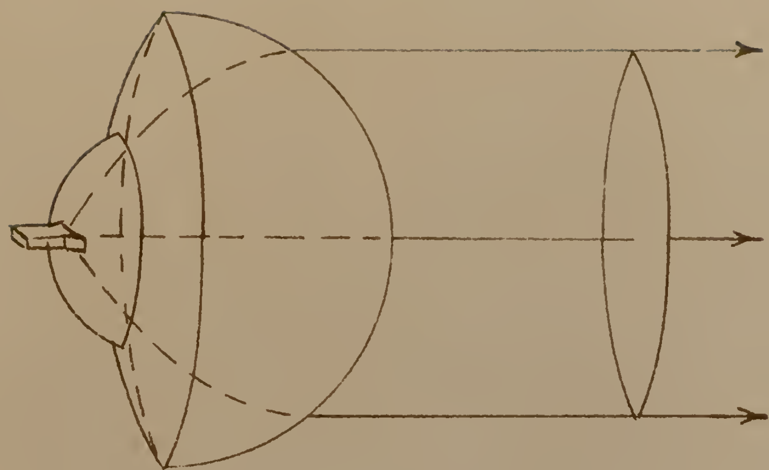

Figure 3.

Spherical Lens with [illegible]

in part "turned in brim" lens [illegible] used [illegible]. Case II was developed wherein a variable refractive index in the [illegible] annulus is adjusted so that n = 1 at the circumference and [illegible] lens is [illegible] as low as possible. The required values of refractive index [illegible] having [illegible] circle [illegible] given [illegible] 4.

Similar results were obtained by [illegible]. Gutman (9) [illegible] perpendicular [illegible] light [illegible]. Gutman also [illegible] to arrive at an expression for the index of refraction

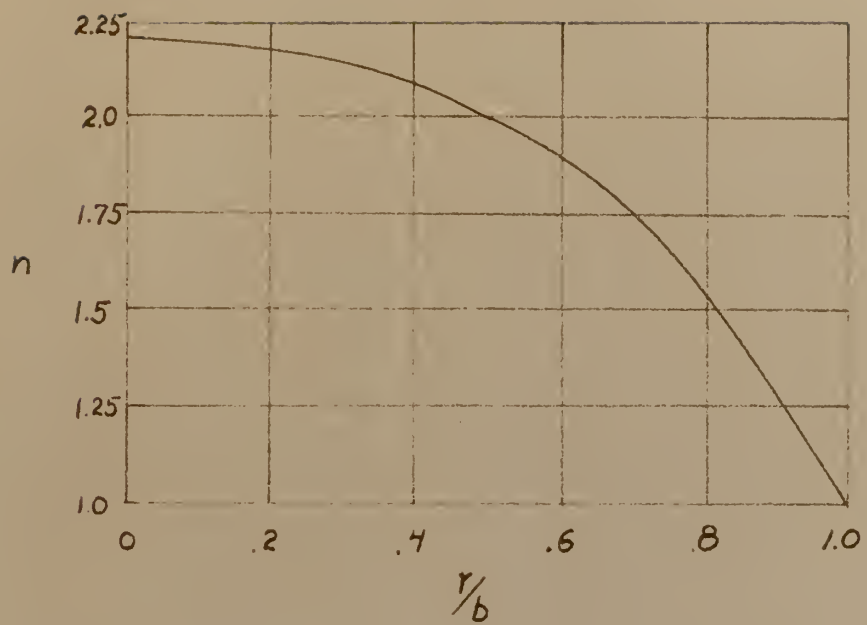

Figure 4.

Refractive Index for Luneberg Lens with Interior Source ($r = b/2$)

$$n = \frac{1}{a}\sqrt{a^2 + b^2 - r^2}$$

This solution utilizes the full aperture of the lens; n is continuous and there are no discontinuities in dn/dr or higher derivatives.

Eaton (7) obtained somewhat different results in that he did not take advantage of full aperture size. For linear phase fronts if

$$n = \sqrt{\frac{2r_0 - r^2}{r_0^2}} \qquad 0 \leqq r \leqq r_0$$

and $$n = \sqrt{\frac{2 - r}{r}} \qquad r_0 \leq r \leq 1$$

the rays would follow elliptical paths so that the effective aperture of the lens would have a diameter

$$D = 2\sqrt{2r_0 - r_0^2}$$

which would limit the beam width obtainable.

He further showed that if the index of refraction of a unit sphere varied as

$$n = r$$

the lens would convert a point source into an apparent line source, Figure 5, or an omnidirectional pattern could be obtained which could be employed for radar beacons, transponders and other services.

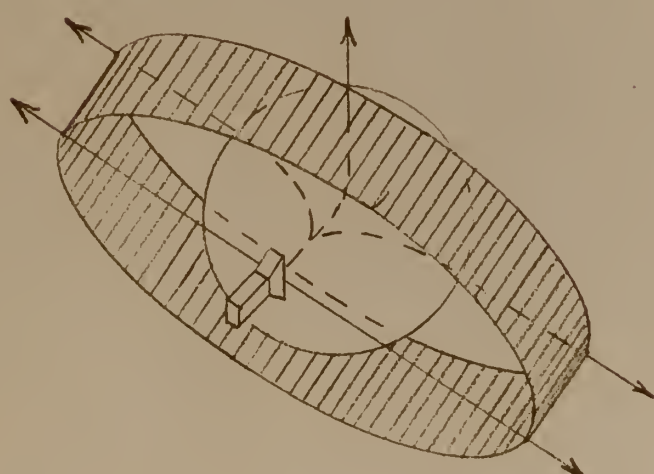

Figure 5.

Spherical Lens that Converts a Point Source Into an Apparent Line Source

If the refractive index varies as

$$n = \sqrt{\frac{2 - r}{r}}$$

the lens would act as an infinite reflecting plane, Figure 6.

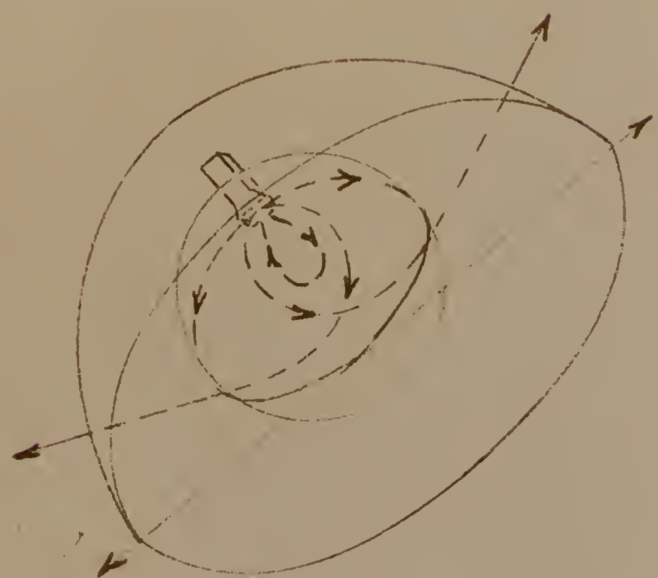

Figure 6.

Spherical Lens that acts as an infinite reflecting plane

CHAPTER II

TWO DIMENSIONAL [illegible]

At the time of this paper technical discussions on only two lenses based on the previous theory were available. Both of these lenses were constructed with application in only two dimensions with the variable index of refraction obtained by spacing variation between plates propagating the TE_{10} mode. The first of these lenses was designed and tested by S. S. D. Jones (1[illegible]) in which the plates were separated by an air dielectric in which the index of refraction varied with the radius as

$$n = n_0 \sqrt{2 - \left(\frac{r}{R} \right)^2}$$

where $n_0 = \sqrt{1 - \left(\frac{\lambda}{2a} \right)^2}$

or < 1. Reflection at the lens aperture ($n_0 \neq 1$) was corrected by the addition of a parallel plate region terminated in a linear aperture to the edge of the lens. This addition maintained symmetry with respect to the diameter rather than the center, resulting in some deterioration and loss of gain off-axis. The lens was designed to operate at a wavelength of 10.0 cm with a 3 ft. aperture; the parallel plate region having a refractive index of $n_c / \sqrt{2}$ where n_c is the refractive index at the center of the lens. Substituting into the above equations gave the lens profile as

$$a = \frac{5}{\sqrt{1 - .372 \left[2 - \left(\frac{r}{4.\text{[illegible]}} \right)^2 \right]}}$$

and spacing in the parallel plate region of 6.32 cm.

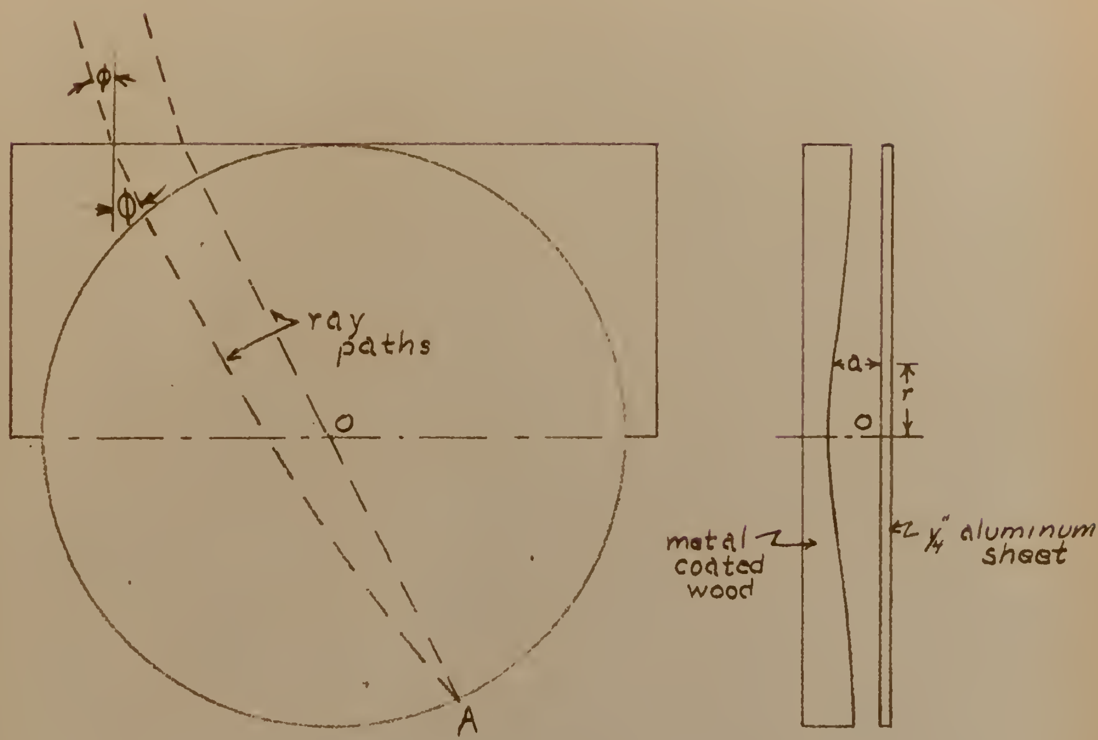

Figure 7.

Construction of Lens
(Not to scale)

The lens was made by spraying a suitably shaped wooden structure with metal as the curved face and a sheet of 1/4 in. aluminum employed as the plane face. The lens feed consisted of (1) a dipole and (2) a length of 10 cm band wave guide, external dimensions 2.4 x 1.5 inches. Resulting radiation patterns show the lack of symmetry and loss of angular deflection as the feed is moved from center. This loss depends on the design of the system and is given by Snell's law

$$\mu = \frac{\sin \phi}{\sin \emptyset}$$

The radiation patterns taken with a dipole feed at central position, 15.75°, 28.1°, and 37.5° (Figure 8) illustrate this loss at a wavelength of 10.7 cm. Figure 9 which employed the waveguide feed, shows the var-

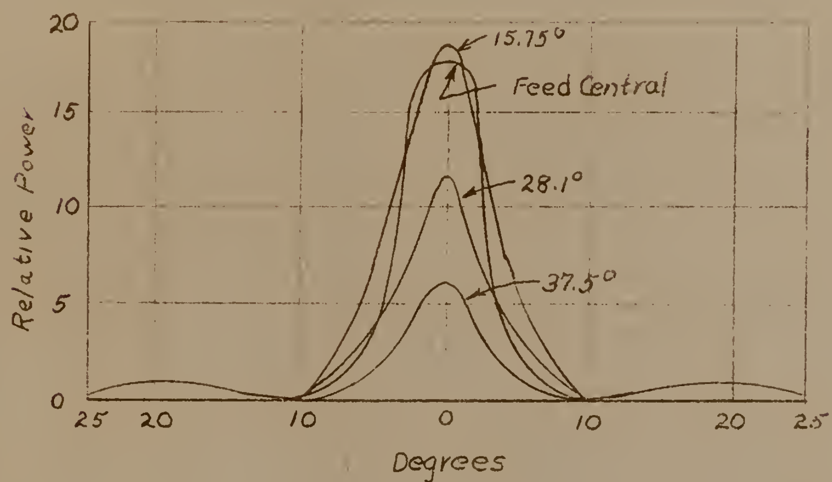

Figure 8.

Radiation Patterns With Dipole Feed

iation of the radiation pattern with change of frequency. The performance of the system agrees well with that predicted by theory. The main limitation is the fairly severe loss of gain for large deflections of the beam from the direction perpendicular to the aperture due to increasing reflection at the straight edge bounding the system.

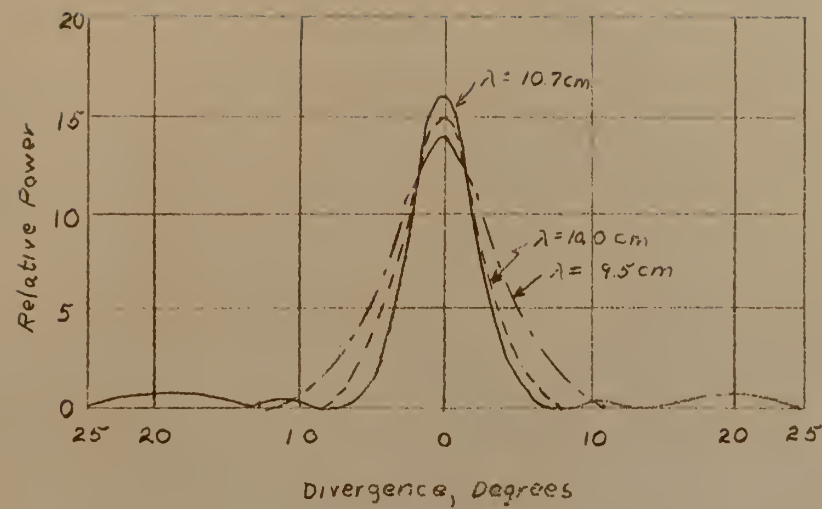

Figure 9.

Radiation Patterns with Waveguide Feed
Feed Central

In order to overcome some of the limitations of the above lens, a similiar construction was employed by G. D. M. Peeler and D. H. Archer (18) except the lens was made of a polystyrene dielectric having a relative dielectric constant K = 2.48. In this design the refractive index within a waveguide $n = \sqrt{K - \left(\frac{\lambda}{2a}\right)^2}$ which when combined with the variable index of refraction required for the Luneberg Lens $n = \sqrt{2 - r^2}$ gives for the resulting required plate spacing

$$a = \frac{\lambda}{2\sqrt{K - 2 + r^2}}$$

which gave a maximum separation of [illegible]" and a separation at the circumference of .[illegible]" for a wavelength of 3.2 cm. Two sheets of 1 x 18 x 36 inch polystyrene were cemented together with Rensselaer polystyrene cement #106 and then machined to give a lens diameter of 36 in. This gave a diameter-to-wavelength ratio of 28.6 which was large enough to produce a reasonable beamwidth. Flanges for attachment of the feed were added over an arc of 1[illegible]° and a half ring on the front of the lens was added to the upper plate so that the dielectric was symmetrically located.

Early patterns taken had an unsymmetrical beam-shape due to slight warping of the polystyrene, causing incorrect plate spacing which was partly corrected by the addition of bolts. The resulting radiation patterns in the E-plane and H-plane are shown in Figures 10 and 11 respectively.

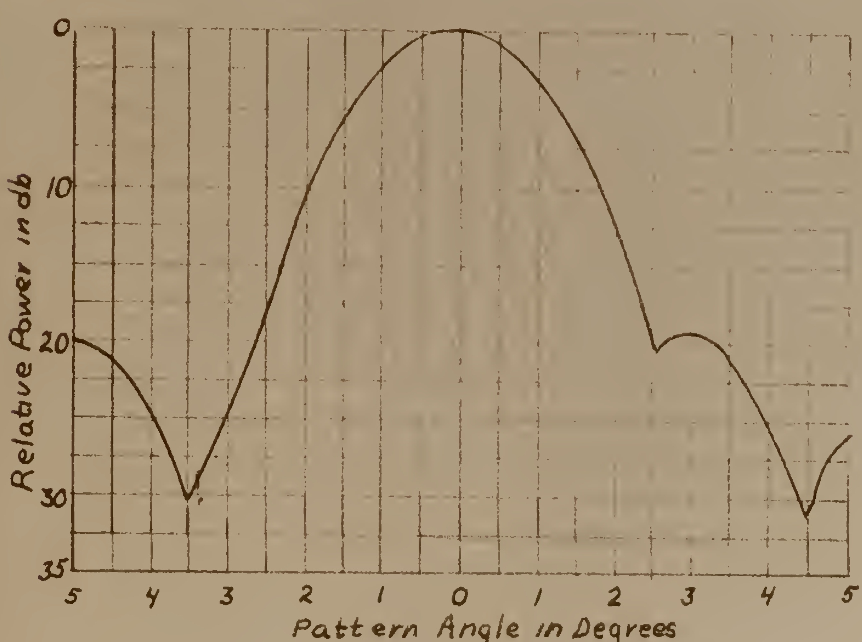

Figure 10.

E-Plane Radiation Pattern

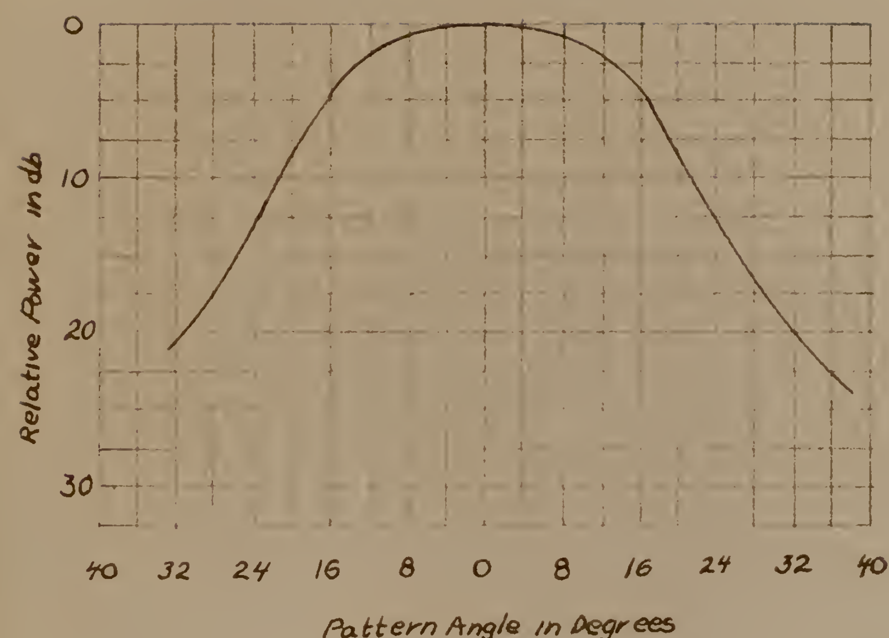

Figure 11.

H-Plane Radiation Pattern

By varying the frequency slight modifications of the E-plane patterns occurred as tabulated below over a 9.6% bandwidth. Equipment was not available for measurements at higher frequencies.

Frequency (Mc)	Beam Width (deg)	Side Lobe Level (db)
8900	3.1	16.9
9000	2.8	15.6
9100	2.5	15.0
9200	2.3	19.9
9300	2.25	17.5
9375	2.25	20.4
9500	2.2	18.8
9600	2.3	19.6
9700	2.4	20.3
9800	2.6	19.7

CHAPTER IV

CONCLUSIONS

From the results obtained through construction and testing of two dimensional models it has been demonstrated that the Luneberg theory of a variable refractive index lens as formulated for optics may be applied to microwave rays in electronics. Such a lens permits focusing without aberrations over a wide angle of scan by movement of feed position alone. Until recent years the problem of attaining a variable refractive index for both optics and microwaves remained unsolved, but now, in electronics, the problem has been solved by the introduction of artificial dielectrics. Through their employment it is now possible to advance to three dimensional models based on the same theory and overcome, to some extent, the problem of wide angle scanning. In addition, the mathematical development, as introduced by Luneberg, has been extended to further reduce some of the problems by development of the variation of refractive index required for reduction of feed circle radius below that of aperture radius. Such a reduction will be limited to the variation of refractive index obtainable with the accompanying losses inherent from propogation through mediums of high refractive index.

Another application, which makes itself apparent, is for aerodynamically suitable external shapes for the aperture. Reflectors of the parabolic shape must be covered with radomes to conform to the aerodynamic requirements for applications on high

speed aircraft. Such radomes in themselves produce losses and distortions in the radiated beam from the antenna. Since the designer, through artificial dielectrics, has control of the wave paths in the lens, it is possible to further extend the theory to external shapes other than portions of a sphere. Such shapes would conform to the requirements and eliminate the use of radomes. In addition, for the same size beam it is possible to reduce the overall external dimensions since, in a Luneberg lens, the aperture would be the diameter of the external dimension while an antenna covered by a radome has an aperture less than the inside diameter of the radome.

Also in aircraft applications wherein the weight of the lens would impose restrictions on its applicability, it is possible to reduce the weight of the lens. Such reduction may be obtained by the employment of a virtual source. Such sources reduce the maximum angle of scan obtainable below 360°. Such sources may be obtained by means of reflectors as under development by G. D. M. Peeler, K. S. Kelleher, and H. P. Coleman (1[illegible]).

At the present time lenses based on the Luneberg theory and constructed of artificial dielectrics are in the development stage. It remains to be seen in future years how effective and useful such lenses may become.

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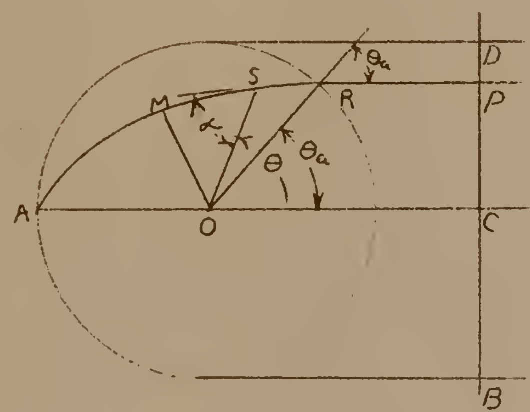
The derivation of an expression for the index of refraction required for focusing on the circumference may be most simply obtained if only a two dimensional system as illustrated in Figure 12 is considered. To obtain an all around scan it is evident that the lens must be symmetrical about its center, O, i.e., the refractive index within the lens must be a function of the radial distance r alone.



$OA = a \quad OS = r \quad OM = r_M$

Figure 12.

Ray Paths in Wide Angle Scanner

The radiated beam will have its maximum in the direction AOC provided that in the free-space region outside the lens the phase of the radiation is constant on lines such as BCD which is perpendicular to AOC. This requires that the electrical path length of any ray from A to the line BD should be constant. Considering a typical ray path ARP then

$$\int_A^R nds + RP = L \qquad (1)$$

where L is independent of the ray path selected. By Fermat's theorem the optical path length between two points is stationary for small variations in the path and this applies to the integral in equation (1). Letting r, θ, be the polar coordinates of a point S on the ray path

$$\int nds = \int n(r) \left[1 + r^2 \theta'^2 \right]^{\frac{1}{2}} dr \qquad (2)$$

where $\theta' = d\theta/dr$ and the refractive index being a function of radius alone is indicated explicitly. The condition that the integral is stationary for small variations in the path is found from the calculus of variations and hence

$$\delta \int n(r) \left[1 + r^2 \theta'^2 \right]^{\frac{1}{2}} dr = 0 \qquad (3)$$

which becomes

$$\int n(r) \frac{r^2 \theta' \delta \theta'}{\left[1 + r^2 \theta'^2 \right]^{\frac{1}{2}}} dr = 0 \qquad (4)$$

Upon integrating by parts

$$n(r) \frac{r^2 \theta' \delta\theta}{\left[1 + r^2 \theta'^2 \right]^{\frac{1}{2}}} \Bigg|_{\theta_1}^{\theta_2} - \frac{d}{dr} \left\{ \frac{n(r) \quad r^2 \theta'}{\left[1 + r^2 \theta'^2 \right]^{\frac{1}{2}}} \right\} \delta\theta \, dr = 0 \qquad (5)$$

and the first term is equal to zero requiring the second term also to be equal to zero or that

$$\frac{n(r) \; r^2 \theta'}{\left[1 + r^2 \theta'^2 \right]^{\frac{1}{2}}} = C \qquad C \geqq 0 \qquad (6)$$

where C is a parameter whose value depends on the ray path selected. From equation (6)

$$r\theta' = \frac{\pm C}{\left[r^2 n^2(r) - C^2\right]^{1/2}} \tag{7}$$

which when integrated gives the equation of the ray path.

Letting α be the angle between the tangent to the ray path and the radius vector r and using the geometric expression

$$r\,\theta' = -\tan\alpha \tag{8}$$

equation (7) becomes

$$rn(r)\sin\alpha = C \tag{9}$$

which must be satisfied for the whole ray path. If the lens is considered as a receiving lens, any rays in free space which are parallel to OC and incident on the lens must pass through the point (a, π).

Designating the minimum distance of the ray path from the origin as r_M, $\alpha = \pi/2$ at M and from equation (9)

$$C = r_M\, n(r_M) \tag{10}$$

Assuming that r n(r) is a monotonic increasing function and integrating equation (7) gives for the ray path within the lens

$$\theta_a - \theta = +\int_r^a \frac{C\,dr}{r\left[r^2 n^2(r) - C^2\right]^{1/2}} \quad \text{when } \theta > \theta_M \tag{11}$$

and

$$\theta_a - \theta = -\int_r^a \frac{C\,dr}{r\left[r^2 n^2(r) - C^2\right]^{1/2}} \quad \text{when } \theta < \theta_M$$

From equation (9)

$$\theta_a = \sin^{-1}\left(\frac{C}{a}\right) \tag{12}$$

since $\alpha = \theta_a$ when $r = a$.

Applying the above conditions to equation (10)

$$\int_{r_M}^{a} \frac{C\,dr}{r\left[r^2 n^2(r) - C^2\right]^{\frac{1}{2}}} = \frac{1}{2}\left[\pi - \sin^{-1}\left(\frac{C}{a}\right)\right] \tag{13}$$

which must be satisfied for all values of C lying between 0 and 1.

Letting

$$\xi = r\,n(r) \tag{14}$$

$$\rho = \log r \tag{15}$$

$$\xi_M = r_M\,n(r_M) = C \tag{16}$$

and

$$\xi_a = a\,n(a) \tag{17}$$

equation (13) may be rewritten as

$$\xi_M \int_{\xi_M}^{\xi_a} \frac{(d\rho/d\xi)\,d\xi}{(\xi^2 - \xi_M^2)^{1/2}} = f(\xi_M) \tag{18}$$

where

$$f(\xi_M) = \frac{1}{2}\left[\pi - \sin^{-1}\left(\frac{\xi_M}{a}\right)\right] \tag{19}$$

if $d\rho/d\xi$ is regarded as the unknown function, equation (19) is of Abel's type and a solution **is** obtained by multiplying both sides of the equation by $\dfrac{1}{(\xi_M^2 - z^2)^{1/2}}$ where z is less than ξ_M and

integrating with respect to ξ_M from z to ξ_a giving

$$\int_z^{\xi_a}\int_{\xi_M}^{\xi_a} \frac{\xi_M (d\rho/d\xi)\, d\xi\, d\xi_M}{\left[(\xi^2 - \xi_M^2)(\xi_M^2 - z^2)\right]^{1/2}} = \int_z^{\xi_a} \frac{f(\xi_M)\, d\xi_M}{(\xi_M^2 - z^2)^{1/2}} \tag{20}$$

The integral on the left may be simplified by interchanging the order of integration and using the result obtained by Luneberg that

$$\int_z^{\xi} \frac{\xi_M \, d\xi_M}{\left[(\xi^2 - \xi_M^2)(\xi_M^2 - z^2)\right]^{1/2}} = \frac{\pi}{2} \tag{21}$$

hence

$$\pi \int_z^{\xi_a} \frac{d\rho}{d\xi}\, d\xi = 2 \int_z^{\xi_a} \frac{f(\xi_M)\, d\xi_M}{(\xi_M^2 - z^2)^{1/2}} \tag{22}$$

or

$$\pi\left[\rho(\xi_a) - \rho(z)\right] = 2 \int_z^{\xi_a} \frac{f(\xi_M)\, d\xi_M}{(\xi_M^2 - z^2)^{1/2}} \tag{23}$$

Since $\rho(\xi_a) = \log_e (a)$ and if $z = r\,n(r)$ then $\rho(z) = \log_e r$ and equation (23) may then be written as

$$\log_e\left(\frac{a}{r}\right) = \frac{2}{\pi} \int_{rn(r)}^{an(a)} \frac{f(\xi_M)\, d\xi_M}{\left[\xi_M^2 - r^2 n^2(r)\right]^{1/2}} \tag{24}$$

which when $f(\xi_M)$ as given in equation (19) is substituted becomes

$$\log_e\left(\frac{a}{r}\right) = \int_{rn(r)}^{an(a)} \frac{d\xi_M}{\left[\xi_M^2 - r^2n^2(r)\right]^{1/2}} - \frac{1}{\pi}\int_{rn(r)}^{an(a)} \frac{\sin^{-1}(\xi_M/a)\,d\xi_M}{\left[\xi_M^2 - r^2n^2(r)\right]^{1/2}} \tag{25}$$

Evaluation of the first integral yields

$$\log_e\left(\frac{a}{r}\right) = \log_e\left[\frac{an(a)}{rn(r)} + \left\{\frac{a^2n^2(a)}{r^2n^2(r)} - 1\right\}^{1/2}\right] + \frac{1}{\pi}\int_{an(a)}^{rn(r)} \frac{\sin^{-1}(\xi_M/a)\,d\xi_M}{\left[\xi_M^2 - r^2n^2(r)\right]^{1/2}} \tag{26}$$

If the lens is normalized so that a is equal to 1 and n(a) matches the refractive index of free space $[n(a) = 1]$ equation (26) becomes

$$\log_e\left(\frac{1}{r}\right) = \log_e\left[\frac{1}{rn(r)} + \left\{\frac{1}{r^2n^2(r)} - 1\right\}^{1/2}\right] + \frac{1}{\pi}\int_{1}^{rn(r)} \frac{\sin^{-1}(\xi_M)\,d\xi_M}{\left[\xi_M^2 - r^2n^2(r)\right]^{1/2}} \tag{27}$$

The integral was shown by Luneberg to be

$$-\tfrac{1}{2}\ \log_e\left[1 + \left\{1 - r^2n^2(r)\right\}^{1/2}\right] \tag{28}$$

which upon substitution yields

$$\log_e n(r) = \frac{1}{2} \log_e \left[1 + \left\{ 1 - r^2 n^2(r \right\}^{1/2} \right] \tag{29}$$

or

$$n^2(r) = 1 + \left\{ 1 - r^2 n^2(r) \right\}^{1/2} \tag{30}$$

giving

$$n(r) = (2 - r^2)^{1/2} \tag{31}$$

as the solution of the problem.